Discrete Mathematics 42 (1982) 221–226
North-Holland Publishing Company

ON THE COMPLEXITY OF A HYPERMAP

A. MACHI'
Istituto Matematico, Università di Roma, 00100 Roma, Italy

Received 6 January 1981
Revised 10 November 1981

It is shown that the number of g-hypertrees spanning the hyperdual $(\sigma^{-1}, \sigma^{-1}\alpha)$ of a hypermap (σ, α) of genus g equals the number of circular permutations ζ such that $g(\sigma, \zeta) = g$ and $g(\alpha, \zeta) = 0$.

1. Introduction

The complexity of a graph is defined as the number of its spanning trees [1]. For planar maps, that is for maps of genus zero, it is known that a map and its dual have equal complexity [2]. In [6] it is proved that a similar result holds for planar hypermaps; this result is a consequence of the fact that the number of codes of a planar hypermap equals the complexity of its hyperdual. This conclusion cannot be drawn if the hypermap is not planar; this is easily shown. In the present paper, the complexity of a hypermap of genus $g \geq 0$ is defined as the number of spanning g-hypertrees of the given hypermap, a g-hypertree being a hypermap of genus g having only one face. The main result will be that there exists a one-to-one correspondence between the set of spanning g-hypertrees of the hyperdual of a given map and a suitable set of circular permutations. The correspondence we set up is of the type used in [6] for the case of a planar hypermap, and the set of circular permutation we consider becomes, in the planar case, the set of codes of [6].

2. Definitions

Definition 1 [3]. A *hypermap* is a pair of permutations (σ, α) of S_n, the symmetric group on n elements, such that the group generated by the two permutations is transitive. If α is an involution without fixed points, the hypermap is a *map*. The cycles of σ, α and $\alpha^{-1}\sigma$ are respectively the *vertices*, *edges* and *faces* of the hypermap. If $\gamma \in S_n$, $z(\gamma)$ denotes the number of cycles of γ, and if $\tau = (i, j)$ is a transposition, then $z(\gamma\tau) = z(\gamma) \pm 1$, according as i and j belong (or not) to the same cycle of γ (Serret's lemma). In the first case we say that τ *disconnects* γ; in the second, that τ *connects* γ.

Definition 2. The genus $g = g(\sigma, \alpha)$ of the hypermap (σ, α) is the nonnegative integer g such that

$$z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma) = n + 2 - 2g.$$

For a proof of the existence of g see [5] or [7]. The hypermap is *planar* if its genus is zero.

Definition 3. A *g-hypertree* is a hypermap of genus g having only one face.

Thus, the hypertrees of [6] are the g-hypertrees with $g = 0$.

Definition 4. Let $\gamma \in S_n$. Then $\theta \in S_n$ is a *refinement* of γ if

$$\gamma = \gamma_1\gamma_2 \cdots \gamma_t, \qquad \theta = \theta_1\theta_2 \cdots \theta_t$$

where:

(i) the γ_i's are the disjoint cycles of γ;
(ii) the θ_i's are products of disjoint cycles;
(iii) γ_i and θ_i operate on the same set of elements;
(iv) $g(\gamma_i, \theta_i) = 0$, $i = 1, 2, \ldots, t$.

It follows from the definition that:

$$z(\theta) + z(\theta^{-1}\gamma) = z(\gamma) + n, \qquad (t = z(\gamma)). \tag{1}$$

Definition 5. The hypermap (σ, α') *spans* the hypermap (σ, α) if α' is a refinement of α.

Definition 6. The *complexity* of a hypermap of genus g is the number of g-hypertrees spanning it.

3. Results

Proposition. *A g-hypertree is of complexity* 1.

Proof. Let (σ, α) be a g-hypertree and let (σ, θ) be a g-hypertree spanning it. From (1) we have

$$z(\theta) + z(\theta^{-1}\alpha) = z(\alpha) + n,$$

and since

$$z(\sigma) + z(\theta) + z(\theta^{-1}\sigma) = n + 2 - 2g,$$

we obtain

$$z(\sigma) + z(\theta) = n + 1 - 2g.$$

As $z(\alpha^{-1}\sigma)=1$ we also have

$$z(\sigma)+z(\alpha)=n+1-2g,$$

and therefore $z(\alpha)=z(\theta)$. Thus $z(\theta^{-1}\alpha)=n$, $\theta=\alpha$ and the g-hypertree (σ,α) only admits itself as a refinement.

Remark. The converse of the above proposition does not hold, as the following example shows. Let $\sigma=(1,3,4,2)$ and $\alpha=(1,3)(2,4)$. The hypermap (σ,α) is planar and has three faces: $(1,4)$, (2) and (3). The permutation α only admits the refinements α, $\alpha'=(1)(3)(2,4)$, $\alpha''=(1,3)(2)(4)$ and $\alpha'''=I$ (identity). All four make up a planar hypermap with σ. Of these, (σ,α) has three faces, (σ,α') and (σ,α'') have two faces. Thus (σ,I) is the only g-hypertree spanning (σ,α).

The following lemma is basic. It has already been used in [5].

Lemma 1. *Let $\langle\sigma,\alpha\rangle$ be a transitive group, σ not a circular permutation. Then:*

(i) *There exists a transposition τ connecting σ and disconnecting α.*

(ii) *For any such transposition the group $\langle\sigma\tau,\alpha\tau\rangle$ is again transitive and the hypermap $(\sigma\tau,\alpha\tau)$ has the same genus as (σ,α).*

Proof. (i) As $z(\sigma)>1$, by transitivity $\alpha\neq 1$, so that there exists a transposition disconnecting α. If all such transpositions also disconnect σ, then the cycles of α are contained in those of σ and $\langle\sigma,\alpha\rangle$ cannot be transitive.

(ii) It is sufficient to prove that if $a=\alpha b$, then there exists $\gamma\in\langle\sigma\tau,\alpha\tau\rangle$ such that $b=\gamma a$. Let $\tau=(i,j)$; since τ disconnects α, α has the form $(\ldots,b,a,\ldots,i,\ldots,j,\ldots)\cdots$ so that $a=\alpha b$. If $a\neq i,j$, then $\gamma=(\alpha\tau)^{-1}$. If $a=i$, then $\alpha=(\ldots,b,i,\ldots,j,\ldots)\cdots$ so that $\alpha\tau=(\ldots,b,j,\ldots)(i,\ldots)\cdots$ Therefore b and j belong to the same cycle of $\alpha\tau$ and since i and j belong to the same cycle of $\alpha\tau$ (τ connects σ) we have $\gamma=(\sigma\tau)^h(\alpha\tau)^k$ for some h and k. Similarly if $a=j$. The last statement follows by calculation.

Lemma 2. *Let $\zeta,\gamma\in S_n$, where ζ is a circular permutation, be such that $g(\zeta,\gamma)=0$, and let τ be a transposition disconnecting γ. Then $g(\zeta,\gamma\tau)=0$.*

Proof. The transitivity of $\langle\zeta,\gamma\tau\rangle$ implies

$$1+z(\gamma\tau)+z(\tau\gamma^{-1}\zeta)\leqslant n+2$$

[5, Lemma 3]. If τ disconnects $\gamma^{-1}\zeta$, then

$$\begin{aligned}1+z(\gamma\tau)+z(\tau\gamma^{-1}\zeta)&=1+1+z(\gamma)+z(\gamma^{-1}\zeta)+1\\&=z(\gamma)+z(\gamma^{-1}\zeta)+3=n+4,\end{aligned}$$

a contradiction. Thus τ connects $\gamma^{-1}\zeta$ and therefore

$$1+z(\gamma\tau)+z(\tau\gamma^{-1}\zeta)=n+2,$$

i.e. $g(\zeta,\gamma\tau)=0$.

Lemma 3. *Let $\gamma \in S_n$ and let*

$$\theta_i = \gamma\tau_1\tau_2 \cdots \tau_i, \qquad (\theta_0 = \gamma)$$

where $\tau_1, \tau_2, \ldots, \tau_i$ are transpositions and τ_i disconnects θ_{i-1}. Then θ_i is a refinement of γ for all i.

Proof. By induction on i. If $i = 0$, then $\theta = \gamma$ is a refinement of itself. Let $i > 0$ and let

$$\gamma = \gamma_1\gamma_2 \cdots \gamma_l, \qquad \theta_{i-1} = \gamma\tau_1\tau_2 \cdots \tau_{i-1} = \gamma'_1\gamma'_2 \cdots \gamma'_l$$

where the γ_j's are the cycles of γ, the γ'_j's operate on the same set as γ_j, $j = 1, 2, \ldots, l$, and $g(\gamma_j, \gamma'_j) = 0$. Since τ_i disconnects θ_{i-1} it is contained in one of the cycles of θ_{i-1}, and so in one of the γ'_j's, let it be γ'_1. Then $\theta_i = (\gamma'_1\tau_i)\gamma'_2 \cdots \gamma'_l$ and, by Lemma 3, $g(\gamma_1, \gamma'_1\tau_1) = 0$.

Definition 7. The *dual* of a hypermap (σ, α) is the hypermap $(\alpha^{-1}\sigma, \alpha^{-1})$ which is obtained from (σ, α) by interchanging faces and vertices and reversing the edges. The *reciprocal* of (σ, α) is the hypermap (α, σ).

Taking dual and reciprocal maps one obtains the following diagram:

$$\begin{tikzcd}[arrows=dash]
 & (\alpha, \sigma) \arrow[r] & (\sigma^{-1}\alpha, \sigma^{-1}) \arrow[dr] & \\
(\sigma, \alpha) \arrow[ur] \arrow[dr] & & & (\sigma^{-1}, \sigma^{-1}\alpha) \\
 & (\alpha^{-1}\sigma, \alpha^{-1}) \arrow[r] & (\alpha^{-1}, \alpha^{-1}\sigma) \arrow[ur] &
\end{tikzcd}$$

These six hypermaps all have the same genus. For a closely related notion ('Tutte's trinity') see [8].

Definition 8. The *hyperdual* of the hypermap (σ, α) is the hypermap $(\sigma^{-1}, \sigma^{-1}\alpha)$.

When (σ, α) is a planar hypermap, there exists a circular permutation ζ such that $g(\sigma, \zeta) = g(\alpha, \zeta) = g(\sigma, \alpha) = 0$ [3]. In the general case, it is proved in [5] that there exists a circular permutation ζ such that:

$$g(\sigma, \zeta) = g(\sigma, \alpha) \quad \text{and} \quad g(\alpha, \zeta) = 0.$$

Let $C_\sigma(\sigma, \alpha)$ be the set of these circular permutations. We can now state and prove the main theorem of this paper.

Theorem. *Let (σ, α) be a hypermap of genus g. Then there exists a one-to-one correspondence between the set of spanning g-hypertrees of the hyperdual $(\sigma^{-1}, \sigma^{-1}\alpha)$ of (σ, α) and the set $C_\sigma(\sigma, \alpha)$. More precisely, let (σ^{-1}, θ) be a spanning g-hypertree of $(\sigma^{-1}, \sigma^{-1}\alpha)$: then $\sigma\theta \in C_\sigma(\sigma, \alpha)$. Conversely, let $\zeta \in C_\sigma(\sigma, \alpha)$. Then $\zeta = \sigma\theta$, where (σ^{-1}, θ) is a spanning g-hypertree of $(\sigma^{-1}, \sigma^{-1}\alpha)$.*

Proof. Let (σ^{-1}, θ) be a spanning g-hypertree of $(\sigma^{-1}, \sigma^{-1}\alpha)$. Then $\sigma\theta$ is a circular permutation and

$$z(\sigma^{-1}) + z(\sigma\theta) + z(\theta) = n + 2 - 2g,$$

so that $g(\sigma, \sigma\theta) = g$. As to $g(\alpha, \sigma\theta)$, by definition, θ is a refinement of $\sigma^{-1}\alpha$ and so applying (1), we obtain

$$n + z(\sigma^{-1}\alpha) = z(\theta) + z(\theta^{-1}\sigma^{-1}\alpha). \tag{2}$$

Moreover,

$$z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma) = n + 2 - 2g, \tag{3}$$

$$z(\sigma) + z(\sigma\theta) + z(\theta) = n + 2 - 2g. \tag{4}$$

It follows that

$$z(\alpha) + z(\sigma^{-1}\alpha) = z(\theta) + 1 \tag{5}$$

and, from (2) and (5), we have

$$\begin{aligned} z(\alpha) + z(\theta^{-1}\sigma^{-1}\alpha) + 1 &= z(\alpha) + n + z(\sigma^{-1}\alpha) - z(\theta) + 1 \\ &= z(\theta) + 1 + n - z(\theta) + 1 = n + 2, \end{aligned}$$

i.e. $g(\alpha, \sigma\theta) = 0$.

Conversely, let $\zeta \in C_\sigma(\sigma, \alpha)$. If α is circular, then $g(\alpha, \zeta) = 0$ implies $\zeta = \alpha$. Taking $\theta = \sigma^{-1}\alpha$ we have that (σ^{-1}, θ) is the required spanning g-hypertree. Let $z(\alpha) > 1$. Since $\langle \alpha, \zeta^{-1}\alpha \rangle$ is transitive, there exists (Lemma 1) a transposition τ_1 connecting α and disconnecting $\zeta^{-1}\alpha$. Similarly, for $p = z(\alpha) - 1$, let $\tau_1, \tau_2, \ldots, \tau_p$ be transpositions such that τ_i connects $\alpha\tau_1\tau_2 \cdots \tau_{i-1}$ and disconnects $\zeta^{-1}\alpha\tau_1\tau_2 \cdots \tau_{i-1}$. Then

$$z(\zeta^{-1}\alpha\tau_1\tau_2 \cdots \tau_p) = z(\zeta^{-1}\alpha) + z(\alpha) - 1.$$

But $g(\alpha, \zeta) = 0$ implies

$$z(\alpha) + z(\zeta) + z(\zeta^{-1}\alpha) = n + 2,$$

so that $\zeta^{-1}\alpha\tau_1\tau_2 \cdots \tau_p = I$, i.e. $\zeta = \alpha\tau_1\tau_2 \cdots \tau_p$.

Let us now show that τ_i disconnects $\sigma^{-1}\alpha\tau_1\tau_2 \cdots \tau_{i-1}$, $i = 1, 2, \ldots, p$, $\tau_0 = 1$. Since $g(\sigma, \zeta) = g$, we have

$$z(\sigma) + 1 + z(\sigma^{-1}\zeta) = n + 2 - 2g,$$

i.e.

$$z(\sigma) + 1 + z(\sigma^{-1}\alpha\tau_1\tau_2 \cdots \tau_p) = n + 2 - 2g,$$

and since $g(\sigma, \alpha) = g$,

$$z(\sigma) + z(\alpha) + z(\sigma^{-1}\alpha) = n + 2 - 2g.$$

Therefore,

$$z(\sigma^{-1}\alpha\tau_1\tau_2 \cdots \tau_p) = z(\sigma^{-1}\alpha) + p.$$

If one of the τ_i connects $\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_{i-1}$, let τ_i be the first such one. Then,

$$z(\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_{i-1}\tau_i)=z(\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_{i-1})-1=z(\sigma^{-1}\alpha)+i-2.$$

But then,

$$z(\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_p)\leq z(\sigma^{-1}\alpha)+i-2+p-i=z(\sigma^{-1}\alpha)+p-2$$

a contradiction.

Thus τ_i disconnects $\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_{i-1}$ for all i, and by Lemma 3 $\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_i$ is a refinement of $\sigma^{-1}\alpha$ for all i. In particular, $\theta=\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_p$ is such a refinement, and the hypermap (σ^{-1},θ) is the required spanning g-hypertree of $(\sigma^{-1},\sigma^{-1}\alpha)$.

Remark. Let $C_\alpha(\sigma,\alpha)$ be the set of circular permutations ζ such that $g(\sigma,\zeta)=0$ and $g(\alpha,\zeta)=g(\sigma,\alpha)=g$. Then a result similar to that of the above theorem holds: *There exists a one-to-one correspondence between the set $C_\alpha(\sigma,\alpha)$ and the set of spanning g-hypertrees of $(\alpha^{-1},\alpha^{-1}\sigma)$, the dual of the hyperdual of (σ,α).*

Remark. The proof of the theorem suggests a way of constructing all the circular permutations of the set $C_\sigma(\sigma,\alpha)$. If α is circular, then α is the unique element of this set. Let $p=z(\alpha)-1$. Then take p transpositions $\tau_1,\tau_2,\ldots,\tau_p$ such that τ_i connects $\alpha\tau_1\tau_2\cdots\tau_{i-1}$ and disconnects $\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_{i-1}$. All elements of $C_\sigma(\sigma,\alpha)$ are of the form $\alpha\tau_1\tau_2\cdots\tau_p$. For the case of a planar hypermap, the same construction is given in [4]. In the same way we see that all spanning g-hypertrees of $(\sigma^{-1},\sigma^{-1}\alpha)$ are of the form $(\sigma^{-1},\sigma^{-1}\alpha\tau_1\tau_2\cdots\tau_p)$.

References

[1] N. Biggs, Algebraic Graph Theory (Cambridge Univ. Press, London, 1974).
[2] N. Biggs, Spanning trees of dual graphs, J. Combin. Theory (B) 11 (1971) 127–131.
[3] R. Cori, Un code pour les graphes planaires et ses applications, Astérisque 27 (1975).
[4] R. Cori, Codage d'une carte planaire et hyperarbres recouvrants, Colloques Internat. C.N.R.S, Orsay (1976).
[5] R. Cori and A. Machì, Su alcune proprietà del genere di una coppia di permutazioni, Boll. Un. Mat. Ital. (5) 18-A (1981) 84–89.
[6] R. Cori and J.G. Penaud, The complexity of a planar hypermap and that of its dual, Annals Discrete Math. 9 (1980) 53–62.
[7] A. Jacques, Sur le genre d'une paire de substitutions, C.R. Acad. Sci. Paris (A) 267 (1968) 625–627.
[8] W.T. Tutte, Duality and trinity, in: A. Hajnal, R. Rado and V.T. Sós, eds., Infinite and Finite Sets Colloq. Math. Soc. Janos Bolyai, Vol. 10 (1975) 1459–1472.